

# A PROOF OF THE EDWARDS-WALSH RESOLUTION THEOREM WITHOUT EDWARDS-WALSH CW-COMPLEXES

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**ABSTRACT.** In the paper titled “Bockstein basis and resolution theorems in extension theory” ([To]), we stated a theorem that we claimed to be a generalization of the Edwards-Walsh resolution theorem. The goal of this note is to show that the main theorem from [To] is in fact equivalent to the Edwards-Walsh resolution theorem, and also that it can be proven without using Edwards-Walsh complexes. We conclude that the Edwards-Walsh resolution theorem can be proven without using Edwards-Walsh complexes.

## 1. INTRODUCTION

In the paper titled “Bockstein basis and resolution theorems in extension theory” ([To]), the following theorem is proven.

**Theorem 1.1.** *Let  $G$  be an abelian group with  $P_G = \mathbb{P}$ , where  $P_G = \{p \in \mathbb{P} : \mathbb{Z}_{(p)} \in \text{Bockstein Basis } \sigma(G)\}$ . Let  $n \in \mathbb{N}$  and let  $K$  be a connected CW-complex with  $\pi_n(K) \cong G$ ,  $\pi_k(K) \cong 0$  for  $0 \leq k < n$ . Then for every compact metrizable space  $X$  with  $X \tau K$  (i.e., with  $K$  an absolute extensor for  $X$ ), there exists a compact metrizable space  $Z$  and a surjective map  $\pi : Z \rightarrow X$  such that*

- (a)  $\pi$  is cell-like,
- (b)  $\dim Z \leq n$ , and
- (c)  $Z \tau K$ .

This theorem turns out to be equivalent to the Edwards-Walsh resolution theorem, first stated by R. Edwards in [Ed], with proof published by J. Walsh in [Wa]:

**Theorem 1.2.** (R. Edwards - J. Walsh, 1981) *For every compact metrizable space  $X$  with  $\dim_{\mathbb{Z}} X \leq n$ , there exists a compact metrizable space  $Z$  and a surjective map  $\pi : Z \rightarrow X$  such that  $\pi$  is cell-like, and  $\dim Z \leq n$ .*

We intend to explain this equivalence in Section 2.

However, the proof of Theorem 1.1 in [To] is interesting because it can be done without using Edwards-Walsh complexes, which were used in the original proof of Theorem 1.2. This requires changing the proof of Theorem 3.9 from [To], which will be done in Section 3 of this paper.

The definition and properties of Edwards-Walsh complexes can be found in [Dr1], [DW] or [KY]. Using Edwards-Walsh complexes, or CW-complexes built similarly to these, was the standard approach in proving resolution theorems, for example in [Wa], [Dr1] and [Le]. But these complexes can become fairly complicated, which also complicates the algebraic topology machinery appearing in proofs using them. The proof of Theorem 1.1, after the

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adjustment of proof of Theorem 3.9 from [To], does not use Edwards-Walsh complexes – instead, it has a more involved point set topological part. Therefore the Edwards-Walsh resolution theorem can be proven without using Edwards-Walsh complexes.

## 2. THE EQUIVALENCE OF THE TWO THEOREMS

We will use the following theorem by A. Dranishnikov, which can be found in [Dr1] as Theorem 11.4, or in [Dr2] as Theorem 9:

**Theorem 2.1.** *For any simple CW-complex  $M$  and any finite dimensional compactum  $X$ , the following are equivalent:*

- (1)  $X\tau M$ ;
- (2)  $X\tau SP^\infty M$ ;
- (3)  $\dim_{H_i(M)} X \leq i$  for all  $i \in \mathbb{N}$ ;
- (4)  $\dim_{\pi_i(M)} X \leq i$  for all  $i \in \mathbb{N}$ .

A space  $M$  is called *simple* if the action of the fundamental group  $\pi_1(M)$  on all homotopy groups is trivial. In particular, this implies that  $\pi_1(M)$  is abelian. Also,  $SP^\infty M$  is the infinite symmetric product of  $M$ , and for a CW-complex  $M$ ,  $SP^\infty M$  is homotopy equivalent to the weak cartesian product of Eilenberg-MacLane complexes  $K(H_i(M), i)$ , for all  $i \in \mathbb{N}$ .

In fact, Theorem 6 from [Dr2] states that if  $X$  is a compact metrizable space, and  $M$  is any CW-complex, then  $X\tau M$  implies  $X\tau SP^\infty M$ . Moreover, since  $SP^\infty M$  is homotopy equivalent to the weak product of Eilenberg-MacLane complexes  $K(H_i(M), i)$ , then  $X\tau SP^\infty M$  implies  $X\tau K(H_i(M), i)$ , for all  $i \in \mathbb{N}$ . This means that the implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) from Theorem 2.1 are true for any compact metrizable space  $X$ , and not just for finite dimensional ones, as well as for any CW-complex  $M$ . So we can restate a part of the statement of Theorem 2.1 in the form we will need:

**Theorem 2.2.** *For any CW-complex  $M$  and any compact metrizable space  $X$ , we have  $X\tau M \Rightarrow X\tau SP^\infty M \Rightarrow \dim_{H_i(M)} X \leq i$  for all  $i \in \mathbb{N}$ .*

Now  $X$  from Theorem 1.1 has property  $X\tau K$ , where  $K$  is a connected CW-complex with  $\pi_n(K) \cong G$ ,  $\pi_k(K) \cong 0$  for  $0 \leq k < n$ , and  $n \in \mathbb{N}$ . By Hurewicz Theorem, if  $n = 1$ , since  $G$  is abelian we get  $H_1(K) \cong \pi_1(K)$ , and if  $n \geq 2$  then  $H_n(K) \cong \pi_n(K)$ . Therefore, by Theorem 2.2,  $X\tau K$  implies  $\dim_{H_n(K)} X \leq n$ , i.e.,  $\dim_G X \leq n$ .

By Bockstein Theorem and basic properties of Bockstein basis, as explained in Lemma 2.4 from [To],  $P_G = \mathbb{P}$  implies that  $\dim_G X = \dim_{\mathbb{Z}} X$ . Now use the Edwards-Walsh resolution theorem to produce a compact metrizable space  $Z$  with  $\dim Z \leq n$ , and a cell-like map  $\pi : Z \rightarrow X$ . Since  $\dim_A Z \leq \dim Z$  for any abelian group  $A$ , using  $A = H_n(K) = G$  as well as other properties of  $K$ , and the fact that  $Z$  is finite dimensional, Lemma 3.10 from [To] shows  $Z\tau K$ .

## 3. HOW TO AVOID USING EDWARDS-WALSH COMPLEXES

In the proof of Theorem 1.1 in [To], the following theorem is used – it appears in [To] as Theorem 3.9. This theorem is a known result, presented in a particular form that was adjusted to fit the needs of the proof of Theorem 1.1. This is why its proof was presented in [To].

**Theorem 3.1** (A variant of Edwards' Theorem). *Let  $n \in \mathbb{N}$  and let  $Y$  be a compact metrizable space such that  $Y = \lim (|L_i|, f_i^{i+1})$ , where  $|L_i|$  are compact polyhedra with  $\dim L_i \leq n + 1$ , and  $f_i^{i+1}$  are surjections. Then  $\dim_{\mathbb{Z}} Y \leq n$  implies that there exists an  $s \in \mathbb{N}$ ,  $s > 1$ , and there exists a map  $g_1^s : |L_s| \rightarrow |L_1^{(n)}|$  which is an  $L_1$ -modification of  $f_1^s$ .*

The proof of this theorem in [To] had two parts, the first part for  $n \geq 2$  and the second for  $n = 1$ . In the first part of the proof, Edwards-Walsh complexes were used. The proof is still correct, but it turns out that there was no need to use Edwards-Walsh complexes. In fact, the entire proof can be simplified, and done for any  $n \in \mathbb{N}$  as it was done for the case when  $n = 1$ . Theorem 3.1 was the only place in [To] where Edwards-Walsh complexes were used, so the main result of [To] can be proven without ever using them. Consequently, the Edwards-Walsh resolution theorem can be proven without using Edwards-Walsh complexes.

The goal of this section is to give a simplified proof for Theorem 3.1. Here is a reminder of some facts from the original paper that are used in the new proof.

First of all, recall that a map  $g : X \rightarrow |K|$  between a space  $X$  and a simplicial complex  $K$  is called a  $K$ -modification of  $f$  if whenever  $x \in X$  and  $f(x) \in \sigma$ , for some  $\sigma \in K$ , then  $g(x) \in \sigma$ . This is equivalent to the following: whenever  $x \in X$  and  $f(x) \in \overset{\circ}{\sigma}$ , for some  $\sigma \in K$ , then  $g(x) \in \sigma$ .

In the course of the simplified proof of Theorem 3.1, we will need the notion of *resolution in the sense of inverse sequences*. This usage of the word resolution is completely different from the notion from the title of this paper. The definition can be found in [MS] for the more general case of inverse systems. We will give the definition for inverse sequences only.

Let  $X$  be a topological space. A *resolution of  $X$  in the sense of inverse sequences* consists of an inverse sequence of topological spaces  $\mathbf{X} = (X_i, p_i^{i+1})$  and a family of maps  $(p_i : X \rightarrow X_i)$  with the following two properties:

- (R1) Let  $P$  be an ANR,  $\mathcal{V}$  an open cover of  $P$  and  $h : X \rightarrow P$  a map. Then there is an index  $s \in \mathbb{N}$  and a map  $f : X_s \rightarrow P$  such that the maps  $f \circ p_s$  and  $h$  are  $\mathcal{V}$ -close.
- (R2) Let  $P$  be an ANR and  $\mathcal{V}$  an open cover of  $P$ . There exists an open cover  $\mathcal{V}'$  of  $P$  with the following property: if  $s \in \mathbb{N}$  and  $f, f' : X_s \rightarrow P$  are maps such that the maps  $f \circ p_s$  and  $f' \circ p_s$  are  $\mathcal{V}'$ -close, then there exists an  $s' \geq s$  such that the maps  $f \circ p_{s'}$  and  $f' \circ p_{s'}$  are  $\mathcal{V}$ -close.

By Theorem I.6.1.1 from [MS], if all  $X_i$  in  $\mathbf{X}$  are compact Hausdorff spaces, then  $\mathbf{X} = (X_i, p_i^{i+1})$  with its usual projection maps  $(p_i : \lim \mathbf{X} \rightarrow X_i)$  is a resolution of  $\lim \mathbf{X}$  in the sense of inverse sequences. Moreover, since every compact metrizable space  $X$  is the inverse limit of an inverse sequence of compact polyhedra  $\mathbf{X} = (P_i, p_i^{i+1})$  (see Corollary I.5.2.4 of [MS]), this inverse sequence  $\mathbf{X}$  will have the property (R1) mentioned above, and we will refer to this property as the *resolution property (R1) in the sense of inverse sequences*.

We will also use stability theory, about which more details can be found in §VI.1 of [HW]. Namely, we will use the consequences of Theorem VI.1. from [HW]: if  $X$  is a separable metrizable space with  $\dim X \leq n$ , then for any map  $f : X \rightarrow I^{n+1}$ , all values of  $f$  are unstable. A point  $y \in f(X)$  is called an *unstable value* of  $f$  if for every  $\delta > 0$  there exists a map  $g : X \rightarrow I^{n+1}$  such that:

- (1)  $d(f(x), g(x)) < \delta$  for every  $x \in X$ , and
- (2)  $g(X) \subset I^{n+1} \setminus \{y\}$ .

Moreover, this map  $g$  can be chosen so that  $g = f$  on the complement of  $f^{-1}(U)$ , where  $U$  is an arbitrary open neighborhood of  $y$ , and so that  $g$  is homotopic to  $f$  (see Corollary I.3.2.1 of [MS]).

Here is a technical result from [To], which is stated there as Lemma 3.7 and used in the proof of Theorem 3.1.

**Lemma 3.2.** *For any finite simplicial complex  $C$ , there is a map  $r : |C| \rightarrow |C|$  and an open cover  $\mathcal{V} = \{V_\sigma : \sigma \in C\}$  of  $|C|$  such that for all  $\sigma, \tau \in C$ :*

- (i)  $\overset{\circ}{\sigma} \subset V_\sigma$ ,
- (ii) if  $\sigma \neq \tau$  and  $\dim \sigma = \dim \tau$ ,  $V_\sigma$  and  $V_\tau$  are disjoint,
- (iii) if  $y \in \overset{\circ}{\tau}$ ,  $\dim \sigma \geq \dim \tau$  and  $\sigma \neq \tau$ , then  $y \notin V_\sigma$ ,
- (iv) if  $y \in \overset{\circ}{\tau} \cap V_\sigma$ , where  $\dim \sigma < \dim \tau$ , then  $\sigma$  is a face of  $\tau$ , and
- (v)  $r(V_\sigma) \subset \sigma$ .

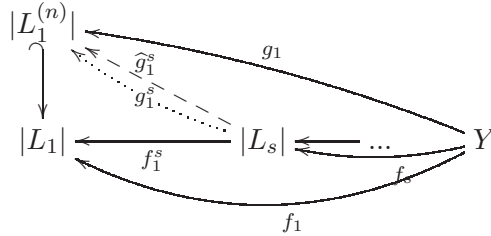
*Simplified proof of Theorem 3.1:* Since  $Y = \lim(|L_i|, f_i^{i+1})$ , where  $|L_i|$  are compact polyhedra with  $\dim L_i \leq n+1$ , we get that  $\dim Y \leq n+1$ . According to Aleksandrov's theorem ([Al]),  $\dim Y$  being finite means  $\dim_{\mathbb{Z}} Y = \dim Y$ . Therefore, assuming  $\dim_{\mathbb{Z}} Y \leq n$  really means that  $\dim Y \leq n$ , too.

Thus we can prove the theorem without using Edwards-Walsh complexes, but instead using the resolution property (R1) in the sense of inverse sequences.

We can construct a map  $g_1 : Y \rightarrow |L_1^{(n)}|$  that equals  $f_1$  on  $f_1^{-1}(|L_1^{(n)}|)$ . This can be done as follows. Let  $\sigma$  be an  $(n+1)$ -simplex of  $L_1$  and  $w \in \overset{\circ}{\sigma}$ . Since  $\dim \sigma = n+1$  and  $\dim Y \leq n$ , the point  $w$  is an unstable value for  $f_1$  ( $f_1$  is surjective, since all our bonding maps  $f_i^{i+1}$  are surjective). Therefore we can find a map  $g_{1,\sigma} : Y \rightarrow |L_1|$  which agrees with  $f_1$  on  $Y \setminus (f_1^{-1}(\overset{\circ}{\sigma}))$ , and  $w \notin g_{1,\sigma}(Y)$ . Then choose a map  $r_\sigma : |L_1| \rightarrow |L_1|$  such that  $r_\sigma$  is the identity on  $|L_1| \setminus \overset{\circ}{\sigma}$  and  $r_\sigma(g_{1,\sigma}(Y)) \cap \overset{\circ}{\sigma} = \emptyset$ . Finally, replace  $f_1$  by  $r_\sigma \circ g_{1,\sigma} : Y \rightarrow |L_1| \setminus \overset{\circ}{\sigma}$ .

Continue the process with one  $(n+1)$ -simplex at a time. Since  $L_1$  is finite, in finitely many steps we will reach the needed map  $g_1 : Y \rightarrow |L_1^{(n)}|$ . Note that from the construction of  $g_1$ , we get

(I)  $g_1|_{f_1^{-1}(|L_1^{(n)}|)} = f_1|_{f_1^{-1}(|L_1^{(n)}|)}$ , and for every  $(n+1)$ -simplex  $\sigma$  of  $L_1$ ,  $g_1(f_1^{-1}(\sigma)) \subset \partial\sigma$ .



Let us choose an open cover  $\mathcal{V}$  of  $|L_1^{(n)}|$  by applying Lemma 3.2 to  $C = L_1^{(n)}$ . Now we can use resolution property (R1) in the sense of inverse sequences: there is an index  $s > 1$  and a map  $\widehat{g}_1^s : |L_s| \rightarrow |L_1^{(n)}|$  such that  $\widehat{g}_1^s \circ f_s$  and  $g_1$  are  $\mathcal{V}$ -close. Define  $g_1^s := r \circ \widehat{g}_1^s : |L_s| \rightarrow |L_1^{(n)}|$ , where  $r : |L_1^{(n)}| \rightarrow |L_1^{(n)}|$  is the map from Lemma 3.2.

Notice that for any  $y \in Y$ , if  $g_1(y) \in \overset{\circ}{\tau}$  for some  $\tau \in L_1^{(n)}$ , then  $g_1(y) \in V_\tau$ , and possibly also  $g_1(y) \in V_{\gamma_j}$ , where  $\gamma_j$  are some faces of  $\tau$  (there can only be finitely many). Then either  $\widehat{g}_1^s \circ f_s(y) \in V_\tau$ , or  $\widehat{g}_1^s \circ f_s(y) \in V_{\gamma_j}$ , for some  $\gamma_j$ . In any case,  $r \circ \widehat{g}_1^s \circ f_s(y) \in \tau$ . Hence,

(II) for any  $y \in Y$ ,  $g_1(y) \in \overset{\circ}{\tau}$  for some  $\tau \in L_1^{(n)}$  implies that  $g_1^s(f_s(y)) \in \tau$ .

Finally, for any  $z \in |L_s|$ ,  $f_s$  is surjective implies that there is a  $y \in Y$  such that  $f_s(y) = z$ . Then  $f_1^s(z) = f_1^s(f_s(y)) = f_1(y)$ . Now  $f_1^s(z)$  is either in  $\overset{\circ}{\sigma}$  for some  $(n+1)$ -simplex  $\sigma$  in  $L_1$ , or in  $\overset{\circ}{\tau}$  for some  $\tau \in L_1^{(n)}$ .

If  $f_1^s(z) \in \overset{\circ}{\sigma}$ , that is  $f_1(y) \in \overset{\circ}{\sigma}$  for some  $(n+1)$ -simplex  $\sigma$ , by (I) we get that  $g_1(y) \in \partial\sigma$ . Then by (II),  $g_1^s(f_s(y)) \in \partial\sigma$ , i.e.,  $g_1^s(z) \in \sigma$ .

If  $f_1^s(z) = f_1(y) \in \overset{\circ}{\tau}$  for some  $\tau \in L_1^{(n)}$ , then (I) implies that  $g_1(y) = f_1(y) \in \overset{\circ}{\tau}$ , so by (II),  $g_1^s(f_s(y)) \in \tau$ , i.e.,  $g_1^s(z) \in \tau$ .

Therefore,  $g_1^s$  is indeed an  $L_1$ -modification of  $f_1^s$ .  $\square$

#### 4. A NOTE ABOUT THE ORIGINAL PROOF OF THE EDWARDS-WALSH RESOLUTION THEOREM

In the original proof of Theorem 1.2 in [Wa], the following theorem is used. It is listed there as Theorem 4.2.

**Theorem 4.1** (R. Edwards). *Let  $n \in \mathbb{N}$  and let  $X$  be a compact metrizable space such that  $X = \lim (P_i, f_i^{i+1})$ , where  $P_i$  are compact polyhedra. The space  $X$  has cohomological dimension  $\dim_{\mathbb{Z}} X \leq n$  if and only if for each integer  $k$  and each  $\varepsilon > 0$  there is an integer  $j > k$ , and a triangulation  $L_k$  of  $P_k$  such that for any triangulation  $L_j$  of  $P_j$  there is a map  $g_k^j : |L_j^{(n+1)}| \rightarrow |L_k^{(n)}|$  which is  $\varepsilon$ -close to the restriction of  $f_k^j$ .*

There were no additional assumptions made about dimension of polyhedra  $P_i$ , so in the proof of this theorem in [Wa], the usage of Edwards-Walsh complexes is indispensable. Therefore, the usage of Edwards-Walsh complexes was necessary in the original proof of Theorem 1.2 in [Wa].

Theorem 3.1 was modeled on Theorem 4.1, but with the additional assumption about dimension of polyhedra  $\dim |L_i| \leq n + 1$ . This assumption, together with  $\dim_{\mathbb{Z}} Y \leq n$  implies that  $\dim Y \leq n$ . Therefore the usage of Edwards-Walsh complexes in its proof can be avoided altogether. In fact, Theorem 3.1 becomes analogous to Theorem 4.1 from [Wa] – a weaker version of Edwards' Theorem:

**Theorem 4.2.** *Let  $n \in \mathbb{N}$  and let  $X$  be a compact metrizable space such that  $X = \lim (P_i, f_i^{i+1})$ , where  $P_i$  are compact polyhedra. The space  $X$  has  $\dim X \leq n$  if and only if for each integer  $k$  and each  $\varepsilon > 0$  there is an integer  $j > k$ , a triangulation  $L_k$  of  $P_k$ , and a map  $g_k^j : P_j \rightarrow |L_k^{(n)}|$  which is  $\varepsilon$ -close to  $f_k^j$ .*

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